1. Two points A and B lie on a smooth horizontal table with $AB = 4a$. One end of a light elastic spring, of natural length a and modulus of elasticity $2mg$, is attached to A. The other end of the spring is attached to a particle P of mass m. Another light elastic spring, of natural length a and modulus of elasticity mg, has one end attached to B and the other end attached to P. The particle P is on the table at rest and in equilibrium.

(a) Show that $AP = \frac{5a}{3}$.

(4)

The particle P is now moved along the table from its equilibrium position through a distance $0.5a$ towards B and released from rest at time $t = 0$. At time t, P is moving with speed v and has displacement x from its equilibrium position. There is a resistance to motion of magnitude $4m\omega v$ where $\omega = \sqrt{\left(\frac{g}{a}\right)}$.

(b) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\omega\frac{\mathrm{d}x}{\mathrm{d}t} + 3\omega^2 x = 0$.

(5)

(c) Find the velocity, $\frac{\mathrm{d}x}{\mathrm{d}t}$, of P in terms of a, ω and t.

(8)
(Total 17 marks)

2. A light elastic spring AB has natural length $2a$ and modulus of elasticity $2mn^2a$, where n is a constant. A particle P of mass m is attached to the end A of the spring. At time $t = 0$, the spring, with P attached, lies at rest and unstretched on a smooth horizontal plane. The other end B of the spring is then pulled along the plane in the direction AB with constant acceleration f. At time t the extension of the spring is x.

(a) Show that $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + n^2x = f.$$

(6)

(b) Find x in terms of n, f and t.

(8)

Hence find

(c) the maximum extension of the spring,

(3)

(d) the speed of P when the spring first reaches its maximum extension.

(2)
(Total 19 marks)

3. A light elastic spring has natural length l and modulus of elasticity mg. One end of the spring is fixed to a point O on a rough horizontal table. The other end is attached to a particle P of mass m which is at rest on the table with $OP = l$. At time $t = 0$ the particle is projected with speed $\sqrt{(gl)}$ along the table in the direction OP. At time t the displacement of P from its initial position is x and its speed is v. The motion of P is subject to air resistance of magnitude $2mv\omega$, where $\omega = \sqrt{\frac{g}{l}}$. The coefficient of friction between P and the table is 0.5.

(a) Show that, until P first comes to rest,

$$\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + \omega^2 x = -0.5g\ .$$

(6)

(b) Find x in terms of t, l and ω.

(6)

(c) Hence find, in terms of ω, the time taken for P to first come to instantaneous rest.

(3)
(Total 15 marks)

4. A small ball is attached to one end of a spring. The ball is modelled as a particle of mass 0.1 kg and the spring is modelled as a light elastic spring *AB*, of natural length 0.5 m and modulus of elasticity 2.45 N. The particle is attached to the end *B* of the spring. Initially, at time $t = 0$, the end *A* is held at rest and the particle hangs at rest in equilibrium below *A* at the point *E*. The end *A* then begins to move along the line of the spring in such a way that, at time *t* seconds, $t \leq 1$, the downward displacement of *A* from its initial position is $2\sin 2t$ metres. At time *t* seconds, the extension of the spring is *x* metres and the displacement of the particle below *E* is *y* metres.

(a) Show, by referring to a simple diagram, that $y + 0.2 = x + 2 \sin 2t$.

(3)

(b) Hence show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 49y = 98\sin 2t$.

(5)

Given that $y = \dfrac{98}{45}\sin 2t$ is a particular integral of this differential equation,

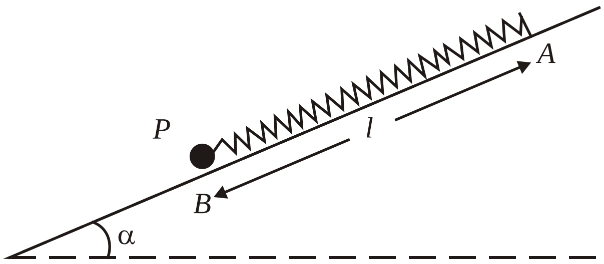
(c) find *y* in terms of *t*.

(5)

(d) Find the time at which the particle first comes to instantaneous rest.

(4)
(Total 17 marks)

5.



A light elastic spring has natural length l and modulus of elasticity $4mg$. One end of the spring is attached to a point A on a plane that is inclined to the horizontal at an angle α, where $\tan \alpha = \frac{3}{4}$. The other end of the spring is attached to a particle P of mass m. The plane is rough and the coefficient of friction between P and the plane is $\frac{1}{2}$. The particle P is held at a point B on the plane where B is below A and $AB = l$, with the spring lying along a line of the plane, as shown in the figure above. At time $t = 0$, the particle is projected up the plane towards A with speed $\frac{1}{2}\sqrt{(gl)}$. At time t, the compression of the spring is x.

(a) Show that $\dfrac{d^2x}{dt^2} + 4\omega^2 x = -g$, where $\omega = \sqrt{\left(\dfrac{g}{l}\right)}$.

(6)

(b) Find x in terms of l, ω and t.

(7)

(c) Find the distance that P travels up the plane before first coming to rest.

(4)

(Total 17 marks)

6. A particle P of mass m is fixed to one end of a light elastic string, of natural length a and modulus of elasticity $2man^2$. The other end of the string is attached to a fixed point O. The particle P is released from rest at a point which is a distance $2a$ vertically below O. The air resistance is modelled as having magnitude $2mnv$, where v is the speed of P.

(a) Show that, when the extension of the string is x,

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + 2n^2x = g$$

(5)

(b) Find x in terms of t.

(7)
(Total 12 marks)

1. (a)

$T_1 = \dfrac{2mge}{a}$; $T_2 = \dfrac{mg(2a-e)}{a}$	B1 (either)	
$T_1 = T_2$		
$2e = (2a - e)$	M1 A1	
$e = \dfrac{2a}{3}$		
$AP = a + \dfrac{2a}{3} = \dfrac{5a}{3}$ * *	A1	4

(b)

$T_2 - T_1 - 4m\omega\dot{x} = m\ddot{x}$		
$\dfrac{mg}{a}\left(\dfrac{4a}{3} - x\right) - \dfrac{2mg}{a}\left(\dfrac{2a}{3} + x\right) - 4m\omega\dot{x} = m\ddot{x}$	M1 A3	
$\ddot{x} + 4\omega\dot{x} + \dfrac{3g}{a}x = 0$		
$\ddot{x} + 4\omega\dot{x} + 3\omega^2 x = 0$ * *	A1	5

(c)

$\lambda^2 + 4\omega\lambda + 3\omega^2 = 0$		
$(\lambda + 3\omega)(\lambda + \omega) = 0$	M1	
$\lambda = -3\omega$ or $\lambda = -\omega$		
$x = Ae^{-\omega t} + Be^{-3\omega t}$	A1	
$\dot{x} = -\omega Ae^{-\omega t} - 3\omega Be^{-3\omega t}$	M1 A1	
$t = 0$, $x = \frac{1}{2}a$, $\dot{x} = 0$	M1	
$\frac{1}{2}a = A + B$		
$0 = -\omega A - 3\omega B$	A1	
$A = \frac{3}{4}a$, $B = -\frac{1}{4}a$	A1	
$\dot{x} = v = \frac{3}{4}a\omega\,(\mathrm{e}^{-3\omega t} - \mathrm{e}^{-\omega t})$	A1	8

[17]

2. (a)

$(\rightarrow)$, T = $m\,\ddot{y}$	M1	
Hooke's Law:		
$T = \dfrac{2mn^2 ax}{2a} = mn^2 x$	B1	
$\left.\begin{array}{l} x + y = \frac{1}{2}ft^2 \\ \dot{x} + \dot{y} = ft \\ \ddot{x} + \ddot{y} = f \end{array}\right\}$	B2	
so, $(\rightarrow)$, $mn^2 x = m\ddot{y} = m(f - \ddot{x})$	DM1	
$\ddot{x} + n^2 x = f$ * *	A1	6

(b)

C.F. : $x = A\cos nt + B\sin nt$	B1	

P.I. : $x = \dfrac{f}{n^2}$	B1	
Gen solution: $x = A\cos nt + B\sin nt + \dfrac{f}{n^2}$	M1	
$\dot{x} = -An\sin nt + Bn\cos nt$ follow their PI	M1 A1ft	
$\left.\begin{array}{l} t = 0, x = 0 \Rightarrow A = -\dfrac{f}{n^2} \\ t = 0, \dot{x} = 0 \Rightarrow B = 0 \end{array}\right\}$	M1 A1	
$x = \dfrac{f}{n^2}\ (1 - \cos nt)$	A1	8

(c)	$\dot{x} = 0 \Rightarrow \quad nt = \pi$	M1	
	$x_{\max} = \dfrac{f}{n^2}(1 - -1) = \dfrac{2f}{n^2}$	M1 A1	3
(d)	$\dot{y} = ft - \dot{x}$	M1	
	$= f\dfrac{\pi}{n} - 0\dfrac{f\pi}{n}$	A1	2

[19]

3.	(a)	$-T - \dfrac{1}{2}mg - 2mv\sqrt{\dfrac{g}{l}} = m\ddot{x}$	M1A3,2,1,0	
		$\dfrac{-mgx}{l} - \dfrac{1}{2}mg - 2m\dot{x}\sqrt{\dfrac{g}{l}} = m\ddot{x}$	M1	
		$\dfrac{d^2x}{dt^2} + 2\omega\dfrac{dx}{dt} + \omega^2 x = -0.5g$ (AG)	A1	6
	(b)	$u^2 + 2\omega u + \omega^2 = 0 \Rightarrow u = \omega$ (twice)		
		CF is $x = e^{-\omega t}(At + B)$	B1	
		PI is $x = -\dfrac{1}{2}l\left(-\dfrac{g}{2\omega^2}\right)$		
		GS is $x = e^{-\omega t}(At + B) - \dfrac{1}{2}l$	M1	
		$t = 0, x = 0 \Rightarrow B = \dfrac{1}{2}l\left(\dfrac{g}{2\omega^2}\right)$	M1	
		$\dfrac{dx}{dt} = -\omega e^{-\omega t}(At + B) + Ae^{-\omega t}$	M1	
		$t = 0,\ \dfrac{dx}{dt} = \sqrt{gl} = \omega l \Rightarrow A = \dfrac{3}{2}\omega l\left(= \dfrac{3\sqrt{gl}}{2}\right)\left(= \sqrt{gl} + \dfrac{0.5g}{\omega}\right)$	M1	

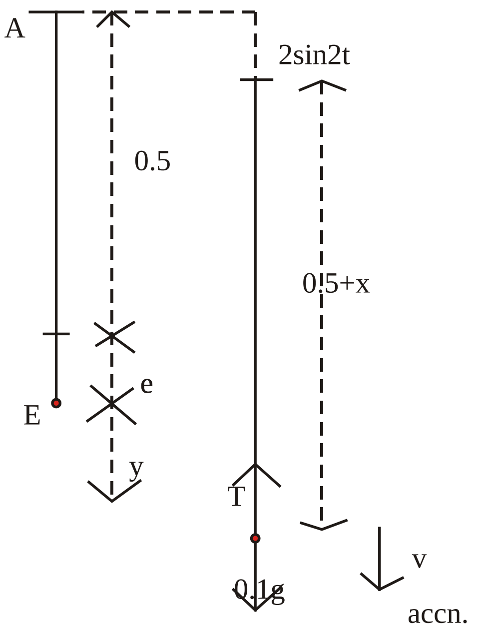
so $x = e^{-\omega t}\left(\frac{3}{2}\omega l t + \frac{1}{2}l\right) - \frac{1}{2}l = \frac{1}{2}le^{-\omega t}(3\omega t + 1) - \frac{1}{2}l$ A1 6

(c) $\frac{dx}{dt} = 0 \Rightarrow -\omega e^{-\omega t}(At + B) + Ae^{-\omega t} = 0$ M1

$\Rightarrow t = \frac{2}{3\omega}$ M1 A1 3

[15]

4. (a)



At E, $\frac{2.45e}{0.5} = 0.1g$ M1

$\Rightarrow e = 0.2$ A1

$\Rightarrow 0.5(\text{or } l) + 0.2 + y = 2 \sin 2t + 0.5 \text{ (or } l) + x$

$\Rightarrow 0.2 + y = 2 \sin 2t + x$ * B1 3

M1 Hooke's law to find extension at equilibrium

A1 cao

B1 Q specifies reference to a diagram.
Correct reasoning leading to **<u>given answer</u>**

(b)	R(↓)		
	$0.1g - T = 0.1\ddot{y}$	M*1	
	$0.1g - \dfrac{2.45x}{0.5} = 0.1\ddot{y}$	M1	
	$0.98 - 4.9(0.2 + y - 2\sin 2t) = 0.1\ddot{y}$ $(-4.9y + 9.8\sin 2t = 0.1\ddot{y})$	DM*1A1	
	$\Rightarrow \dfrac{d^2y}{dt^2} + 49y = 98\sin 2t$ *	A1cso	5

M1 Use of F = ma. Weight, tension and acceleration. Condone sign errors.

M1 Substitute for tension in terms in terms of x

M1 Use given result to substitute for x in terms of y

A1 Correct unsimplified equation

A1 Rearrange to **<u>given form</u>** cso.

(c)	CF is $y = A\cos 7t + B\sin 7t$	M1	
	Hence GS is $y = A\cos 7t + B\sin 7t + \dfrac{98}{45}\sin 2t$	A1	
	$t = 0, y = 0$: $0 = A$ so, $y = B\sin 7t + \dfrac{98}{45}\sin 2t$	B1	
	$\dot{y} = 7B\cos 7t + \dfrac{196}{45}\cos 2t$	M1	
	$t = 0,\ \dot{y} = 0 : 0 = 7B + \dfrac{196}{45} \Rightarrow B = -\dfrac{28}{45}$	M1	5
	$\Rightarrow y = \dfrac{14}{45}(7\sin 2t - 2\sin 7t)$		

M1 Correct form for CF

A1 GS for y correct

B1 Deduce coefficient of cos $\theta = 0$

M1 Differentiate their y and substitute $t = 0,\ \dot{y} = 0$

A1 y in terms of t. Any exact equivalent.

(d) $\dot{y} = \frac{14}{45}(14\cos 2t - 14\cos 7t)$ B1

$\dot{y} = 0 \Rightarrow \cos 2t = \cos 7t$ M1

$\Rightarrow 7t = 2k\pi \pm 2t$ M1

$k = 1 \Rightarrow 9t = 2\pi$ (or $5t = 2\pi$)

$t = \frac{2\pi}{9}$, accept 0.698s, 0.70s. A1 4

B1 $\dot{y}$ correct

M1 set $\dot{y} = 0$

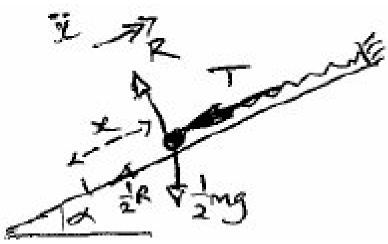
M1 solve for general solution for t: $7t = 2k\pi \pm 2t$

or: $\sin\frac{9t}{2} \times \sin\frac{5t}{2} = 0 \Rightarrow \sin\frac{9t}{2} = 0$ or $\sin\frac{5t}{2} = 0$

A1 Select smallest value

[17]

5. (a)



$F = \frac{1}{2}R$ M1

$R = mg\cos\alpha$ B1

$T = \frac{4mgx}{l}$ B1

(↗): $-F - mg\sin\alpha - T = m\ddot{x}$ M1 A1

$-\frac{1}{2}.\frac{4}{5}mg - \frac{3}{5}mg - \frac{4mgx}{l} = m\ddot{x}$

$\Rightarrow \frac{d^2x}{dt^2} + 4w^2x = -g$ A1 6

(b) $\left(w = \sqrt{\frac{s}{L}}\right)$

$m^2 + 4w^2 = 0 \quad \Rightarrow m = \pm 2wi$

C.F. $x = A\sin 2wt + B\cos 2wt$ M1

P.I. $x = \frac{-g}{4w^2} = \frac{-l}{4}$ B1

G.S. $x = A\sin 2wt + B\cos 2wt - \frac{l}{4}$ B1

$t = 0, x = 0 \qquad B = \frac{l}{4}$

$\dot{x} = 2wA\cos 2wt - 2wB\sin 2wt$ M1 A1

$t = 0,\ \dot{x} = \frac{1}{2}\sqrt{gl} : \frac{\sqrt{gl}}{2} = 2\sqrt{\frac{g}{l}}A \Rightarrow A = \frac{1}{4}$ M1

$\Rightarrow x = \frac{l}{4}(\sin 2wt + \cos 2wt - 1)$ A1 7

(c) $\dot{x} = 0 \Rightarrow \cancel{2w}A\cos 2wt - \cancel{2w}B\sin 2wt = 0$ M1

$\Rightarrow \tan 2wt = \frac{A}{B} = 1$

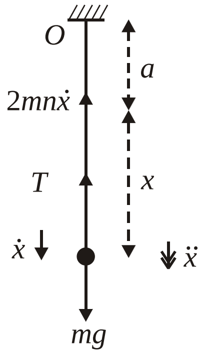
$\Rightarrow 2wt = \frac{\pi}{4}$ (first value) A1

$\Rightarrow x = \frac{1}{4}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1)$ M1

$= \frac{l}{4}(\sqrt{2} - 1)$ A1 4

[17]

6. (a)



$mg - T - 2mn\dot{x} = m\ddot{x}$ M1 A1 A1

$mg - \dfrac{2man^2x}{a} - 2mn\dot{x} = m\ddot{x}$ M1

$\ddot{x} + 2n\dot{x} + 2n^2x = g$ (*) A1 5

(b) AE: $u^2 + 2nu + 2n^2 = 0$ M1

$(u + n)^2 = -n^2$

$u = -n \pm n\text{i}$ A1

CF: $x = e^{-nt}(A\cos nt + B\sin nt)$, PI: $x = \dfrac{g}{2n^2}$ M1

GS: $x = e^{-nt}(A\cos nt + B\sin nt) + \dfrac{g}{2n^2}$ A1

$t = 0, x = a, \dot{x} = 0$: $A = a - \dfrac{g}{2n^2}$ M1

$\dot{x} = e^{-nt}(-An\sin nt + Bn\cos nt) - ne^{-nt}(A\cos nt + B\sin nt)$ M1

$x = e^{-nt}\left(a - \dfrac{g}{2n^2}\right)(\cos nt + \sin nt) + \dfrac{g}{2n^2}$ A1 7

[12]

1. Part (a) was answered well, and candidates tended to be successful in applying Hooke's law and equating the tensions in order to find the distance *AP* at equilibrium.

In part (b) clear diagrams showing the point from which x was measured were an advantage here and were often lacking, resulting in tension being in the wrong directions and/or with the wrong magnitudes. With x defined in the question, candidates often had difficulty in obtaining correct expressions for the extension in each string in terms of x and fudges aimed at obtaining the given differential equation were very common. Some candidates struggled with the mechanics involved and did not include all the relevant terms in setting up their equation of motion.

In part (c) a great many candidates were able to solve the second order differential equation correctly, demonstrating a sound grasp of the pure mathematics involved, although a few were expecting a trigonometric solution and forced their auxiliary equation to produce one. Many candidates were able to use correct boundary conditions to find the unknowns and give a correct final result. The use of a or 0 in place of $0.5a$ for the initial displacement of P was surprisingly common. A few candidates overlooked the request to give the velocity of P as their final answer.

2. In part (a) many candidates claimed to have derived the given equation, but very few actually did so correctly having considered the equation of motion of the particle P. Those candidates who started with a clear diagram were far more likely to realise that they needed to consider both the distance moved by P and the extension in the spring. It was very common for candidates to score only one mark here, for finding the tension in the spring correctly.

Almost all candidates in part (b) were confident in attempting to solve the second order differential equation, although several did not choose a correct form for the complementary function, and a few struggled with the particular integral. It was common to see the correct solution for x.

In part (c) it was reassuring to find many candidates knowing that the maximum value of $(1 - \cos nt)$ is 2, although 1 was a popular alternative answer. Candidates who did not use the basic properties of the trig function were often able to find the maximum extension by using calculus, but here too there were some difficulties in identifying the value(s) of t for which $\sin nt = 0$.

The response to part (d) confirmed that many candidates had little or no idea of the correct derivation of the equation in (a). Most candidates believed that the speed of P and the rate of change of the extension in the spring were equal. Correct responses were seen, but usually only from the stronger candidates.

3. Establishing the equation of motion of the system was usually quite well done, although fuller and more convincing explanations should have been given in many cases.
Solving the differential equation proved more taxing. There were frequent errors in the complementary function, the addition of the particular integral, and the stage in the solution at which arbitrary constants should be found. Few candidates paid attention to the dimensional consistency of their equations. The realisation that x is a length and that the particular integral could also be expressed as a length only ($-l/2$) would have simplified many solutions and enabled x to be found in terms of l, w and t only. Those who did obtain the correct general solution tended to be successful in using the initial conditions to find the constants. Some candidates with complicated expressions for x were able to simplify their final answer to $\frac{2}{3\omega}$ but many were content to give a final answer that was dimensionally inconsistent with a time.

4. (a) Most candidates started by finding the correct value for the equilibrium extension. Those who were able to obtain the given result convincingly initially included the natural length in their equation. The question did ask candidates to refer to a simple diagram in their explanation – some of the diagrams suggested that candidates had little appreciation of the situation described in the question.

(b) A significant number of candidates omitted this part of the question and others tried to obtain the given result by differentiating the relationship obtained in part (a). Those who began with Newton's Second Law, with the correct acceleration, generally scored full marks for this part although it was quite common to see confusion between $\ddot{x}$ and $\ddot{y}$ which was fudged to obtain the required result.

(c) Candidates were usually able to give the correct form of the general solution and then find values for the constants of integration by correct methods. Candidates who used only the complementary function in their working to find the values of the two constants scored little or nothing for the remainder of the question. Similarly, there were some candidates who did not obtain the correct form for the complementary function who could not then score many further marks.. Some candidates attempted to use incorrect initial conditions based on $t = 0$, $x = 0$, $\dot{x} = 0$, and using the result from part (a). Several candidates created additional work for themselves by working from the general form of the particular integral to deduce the given form, others substituted the given term into the differential equation to show that it works.

(d) Most candidates differentiated their solution to part (c) and put it equal to 0. Many candidates obtained the correct trigonometric equation cos $2t$ = cos $7t$. The majority of candidates did not know how to solve this equation and stopped at this point. Those who were able to solve it generally used the correct sum/product formula, rather than the general solution for the cosine function.

5. (a) Most candidates were able to make a reasonable attempt although there were some sign errors in Newton's second law. Some weaker candidates missed out the component of the weight.

(b) The auxiliary equation was usually solved correctly. Subsequently a common error was either not to find a particular integral or to attempt to find it having already used the initial conditions on the complementary function – this was heavily penalised.

(c) Although the straightforward method of equating the velocity to zero was usually known, those candidates who had not simplified their answer to part (b) were often unable to complete this part. There were a very small number who attempted to use an energy method, occasionally correctly.

6. No Report available for this question.